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DEMAZURE EMBEDDINGS ARE SMOOTH

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Abstract. We prove the conjecture of M. Brion stating that the closure of the orbit of a selfnormalizing spherical subalgebra in the corresponding Grassmanian is smooth.

1. Introduction

Throughout the paper the base field $\mathbb{K}$ is algebraically closed and of characteristic zero. Let G be a connected semisimple algebraic group of adjoint type and $\mathfrak{g}$ its Lie algebra.

A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is said to be *spherical* if there is a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ such that $\mathfrak{b} + \mathfrak{h} = \mathfrak{g}$. For example, if $\mathfrak{h} = \mathfrak{g}^\sigma$ for an involutory automorphism σ of $\mathfrak{g}$ (in this case $\mathfrak{h}$ is called symmetric), then $\mathfrak{h}$ is spherical, see [V]. For a symmetric subalgebra $\mathfrak{h} \subset \mathfrak{g}$ De Concini and Procesi proved in [CP] that the closure $\overline{G\mathfrak{h}}$ of the orbit $G\mathfrak{h}$ in the corresponding Grassman variety is smooth. Earlier this fact in a few special cases was proved by Demazure, [D]. It was conjectured by Brion in [Br] that the same is true for any spherical subalgebra $\mathfrak{h}$ coinciding with its normalizer (note that a symmetric subalgebra satisfies this condition). The variety $\overline{G\mathfrak{h}}$ is called the *Demazure embedding* of $G\mathfrak{h}$.

Let us explain why the smoothness of Demazure embeddings is important. There is a nice class of smooth projective G-varieties, so called *wonderful varieties*, possessing many amazing properties, see [T], Section 30, for a review. Knop proved in [K] that a homogeneous space $G\mathfrak{h}$ is embedded as an open G-orbit into (a unique) wonderful variety provided $\mathfrak{h}$ is spherical and coincides with its normalizer. If $\overline{G\mathfrak{h}}$ is smooth, then it coincides with this wonderful variety. Brion's results, [Br], imply that the normalization of $\overline{G\mathfrak{h}}$ is wonderful.

The following theorem is the main result of this paper.

Theorem 1. *Let $\mathfrak{h}$ be a spherical subalgebra of $\mathfrak{g}$ coinciding with its normalizer. Then the Demazure embedding $\overline{G\mathfrak{h}}$ is smooth.*

Let us note that this theorem was already proved under certain restrictions on $\mathfrak{g}$. In [Lu3] Luna gave the proof in the case when all simple ideals of $\mathfrak{g}$ are $\mathfrak{sl}$'s. In [BP] Luna's technique was extended to the case when any simple ideal of $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_k$ or $\mathfrak{so}_{2k}$. In this paper we do not follow Luna's approach directly, however we use many results from [Lu3].

Theorem 1 is proved in Section 3. In Section 2 we recall some previosuly obtained results related to wonderful varieties.

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2. Preliminaries

Below G is a connected semisimple algebraic group of adjoint type, B its Borel subgroup, T a maximal torus of B, Π is the system of simple roots of G and B^- is the Borel subgroup of G containing T and opposite to B. Let $\mathfrak{X}(T)$ denote the character lattice of T (=the root lattice of G).

At first, let us recall the definition of a wonderful variety. References are [Lu1],[Lu2],[Lu3], [T], Section 30.

Definition 2.1. A G-variety X is called *wonderful* if the following conditions are satisfied:

(1) X is smooth and projective.
(2) There is an open G-orbit $X^0 \subset X$.
(3) $X \setminus X^0$ is a divisor with normal crossings.
(4) Let $D_1, \ldots, D_r$ be irreducible components of $X \setminus X^0$. Then for any subset $I \subset \{1, \ldots, r\}$ the subvariety $\bigcap_{i \in I} D_i \setminus \cup_{j \notin I} D_j$ is a single G-orbit.

The number r is called the rank of X and is denoted by $\mathrm{rk}_G(X)$.

Note that $\bigcap_{i \in I} D_i$ is a wonderful G-variety of rank $\#I$ for any $I \subset \{1, \ldots, r\}$.

It is known that a wonderful variety X is spherical, that is B has an open orbit on X.

Let us now establish some combinatorial invariants of a wonderful G-variety X.

Note that $\cap_{i=1}^r D_r$ is a generalized flag variety. So there is a unique B^--stable point $z \in \cap_{i=1}^r D_r$. The group T acts linearly on the normal space $T_zX/T_z(Gz)$. Note that $T_zX/T_z(Gz) = \bigoplus_{i=1}^r T_zX/T_zD_i$. Let α_i denotes the character of the action $T : T_zX/T_zD_i$. Let us fix an $N_G(T)/T$-invariant scalar product on $\mathfrak{t}^*(\mathbb{Q})$. With respect to this scalar product $\Psi_{G,X} := \{\alpha_1, \ldots, \alpha_r\}$ is a system of simple root of some root system in the linear span of $\alpha_1, \ldots, \alpha_r$. The set $\Psi_{G,X}$ is called the *system of spherical roots* of X. The definition of $\Psi_{G,X}$ given here agrees with that we use in [Lo2]. Let $\mathfrak{X}_{G,X}$ denote the sublattice in $\mathfrak{X}(T)$ generated by $\Psi_{G,X}$. It is known that $\mathfrak{X}_{G,X}$ coincides with the set of all $\lambda \in \mathfrak{X}(T)$ such that there is a B-semiinvariant rational function f_λ on X of weight λ (determined uniquely up to rescaling because X is spherical). Choose a subset $\Psi_0 \subset \Psi_{G,X}$. Put $\overline{X}^{\alpha_i} := D_i, \overline{X}^{\Psi_0} := \cap_{\alpha \in \Psi_0} \overline{X}^{\alpha}, X^{\Psi_0} = \overline{X}^{\Psi_0} \setminus \cup_{\alpha \notin \Psi_0} \overline{X}^{\alpha}$.

Let $\mathcal{D}_{G,X}$ denote the set of all prime B-stable but not G-stable divisors on X (this definition differs slightly from that used in [Lo2]). To each $D \in \mathcal{D}_{G,X}$ we assign its stabilizer $G_D \subset G$, which is a parabolic subgroup of G containing B, and an element $\varphi_D \in \mathfrak{X}^*_{G,X}$ defined by $\langle \varphi_D, \lambda \rangle = \mathrm{ord}_D(f_\lambda)$. For $\alpha \in \Pi$ by P_α we denote the minimal parabolic subgroup of G containing B corresponding to the simple root α. Put $\mathcal{D}_{G,X}(\alpha) = \{D \in \mathcal{D}_{G,X} | P_\alpha \not\subset G_D\}$.

The following propositions are due to Luna, see [Lu1],[Lu2].

Proposition 2.2. *For $\alpha \in \Pi(\mathfrak{g})$ exactly one of the following possibilities takes place:*

() $\mathcal{D}_{G,X}(\alpha) = \varnothing$.
(b) $\alpha \in \Psi_{G,X}$. *Here* $\mathcal{D}_{G,X}(\alpha) = \{D^+, D^-\}$ *and* $\varphi_{D^+} + \varphi_{D^-} = \alpha^\vee|_{\mathfrak{a}_{G,X}}, \langle \varphi_{D^\pm}, \alpha\rangle = 1$.
(c) $2\alpha \in \Psi_{G,X}$. *In this case* $\mathcal{D}_{G,X}(\alpha) = \{D\}$ *and* $\varphi_D = \frac{1}{2}\alpha^\vee|_{\mathfrak{a}_{G,X}}$.
(d) $\mathbb{Q}\alpha \cap \Psi_{G,X} = \varnothing, \mathcal{D}_{G,X}(\alpha) \neq \varnothing$. *In this case* $\mathcal{D}_{G,X}(\alpha) = \{D\}$ *and* $\varphi_D = \alpha^\vee|_{\mathfrak{a}_{G,X}}$.

We say that a root $\alpha \in \Pi$ is *of type* a) (or b),c),d)) if the corresponding possibility takes place for α.

Proposition 2.3. *Let $\alpha, \beta \in \Pi(\mathfrak{g})$. If $\mathcal{D}_{G,X}(\alpha) \cap \mathcal{D}_{G,X}(\beta) \neq \varnothing$, then exactly one of the following possibilities takes place:*

(1) *α, β are of type b) and $\#\mathcal{D}_{G,X}(\alpha) \cap \mathcal{D}_{G,X}(\beta) = 1$.*
(2) *α, β are of type d), $\langle \alpha^\vee, \beta\rangle = 0, \alpha^\vee - \beta^\vee|_{\mathfrak{a}_{G,X}} = 0$, and $\alpha + \beta = \gamma$ or 2γ for some $\gamma \in \Psi_{G,X}$.*

Conversely, if $\alpha, \beta \in \Pi$ are such as in (2), then $\mathcal{D}_{G,X}(\alpha) = \mathcal{D}_{G,X}(\beta)$.

Proposition 2.4. *Let $\alpha \in \Psi_{G,X}, \beta \in \Pi \cap \Psi_{G,X}, D \in \mathcal{D}_{G,X}(\beta)$. Then $\langle \varphi_D, \alpha\rangle \leqslant 1$ and the equality holds iff $\alpha \in \Pi, D \in \mathcal{D}_{G,X}(\alpha)$.*

Proof. It follows from results of [Lu1], Subsection 3.5, (see also [Lu2], Subsection 3.2) that in the proof one may replace X with $\overline{X}^{\alpha,\beta}$. In this case everything follows from the classification in [W]. ☐

Now we are going to describe the localization procedure for wonderful varieties.

Choose a subset $\Pi' \subset \Pi$. Let M be the Levi subgroup of G corresponding to Π'. Put $G_{\Pi'} := (M, M), Q^- = B^- M$. Then there is a $G_{\Pi'}$-stable subvariety $X_{\Pi'} \subset X^{\mathrm{R}(Q^-)}$ (where $\mathrm{R}(\cdot)$ denotes the radical) satisfying the following conditions:

(1) $z \in X_{\Pi'}$.
(2) $X_{\Pi'}$ is a wonderful $G_{\Pi'}$-variety.
(3) $\Psi_{G_{\Pi'}, X_{\Pi'}} = \{\alpha \in \Psi_{G,X} | \operatorname{Supp}(\alpha) \subset \Pi'\}$ (here and below $\operatorname{Supp}(\alpha)$ stands for the set of all $\beta \in \Pi$ such that the coefficient of β in α is nonzero).
(4) For any $\alpha \in \Pi'$ there is a bijection $\iota : \mathcal{D}_{G_{\Pi'}, X_{\Pi'}}(\alpha) \to \mathcal{D}_{G,X}(\alpha)$ such that φ_D is the projection of $\varphi_{\iota(D)}$ to $\mathfrak{X}_{G_{\Pi'}, X_{\Pi'}}$.
(5) $GX_{\Pi'} = \overline{X}^{\Psi_{G,X} \setminus \Psi_{G_{\Pi'}, X_{\Pi'}}}$.

The $G_{\Pi'}$-variety $X_{\Pi'}$ is called the *localization* of X at Π'.

Proceed to the definition of Demazure morphisms.

Choose a point $x \in X^\varnothing$ and put $\mathfrak{h} := \mathfrak{g}_x, d := \dim \mathfrak{h}$. The Demazure morphism $\delta_X : X \to \operatorname{Gr}_d(\mathfrak{g})$ is defined as follows: it maps $y \in X$ to the inefficiency kernel of the representation of $\mathfrak{g}_y$ in $T_x X / T_x(Gy)$, for example, $\delta_X(x) = \mathfrak{h}, \delta_X(z)$ is the intersection of the kernels of all $\alpha \in \Psi_{G,X}$, where $\alpha \in \Psi_{G,X}$ is considered as a character of a parabolic subalgebra $\mathfrak{g}_z$. It is known that the image of $\mathfrak{g}_y$ in $\mathfrak{gl}(T_x X / T_x(Gy))$ is a Cartan subalgebra, so $\operatorname{im} \delta_X$ does lie in $\operatorname{Gr}_d(\mathfrak{g})$. Moreover, $\operatorname{im} \delta_X = \overline{G\mathfrak{h}}$. In [Br] Brion proved that δ_X is the normalization

morphism. So $\overline{G\mathfrak{h}}$ is smooth iff δ_X is an isomorphism. Further, Brion's result implies the following statement.

Lemma 2.5. *Any element of* $\operatorname{im}\delta_X$ *is a spherical algebraic subalgebra of* G.

It is known that $\mathfrak{n}_\mathfrak{g}(\mathfrak{h})=\mathfrak{h}$. Conversely, as Knop proved in [K], for any spherical subalgebra $\mathfrak{h}$ coinciding with its normalizer there is a wonderful variety X such that $X^\varnothing\cong G/N_G(\mathfrak{h})$. Note that $X^\varnothing = G/N_G(\mathfrak{h})$ has no nontrivial equivariant automorphisms. We say that a wonderful variety X is *rigid* if $X^\varnothing$ has no nontrivial equivariant automorphisms. So theorem 1 is equivalent to the claim that δ_X is an isomorphism provided X is rigid.

The rigidity of X can be expressed in terms of $\Psi_{G,X},\mathcal{D}_{G,X}$. To state this result we need the following definition.

Definition 2.6. An element $\alpha\in\Psi_{G,X}$ is said to be *distinguished* if one of the following conditions holds:

(1) $\alpha\in\Pi$ and $\varphi_{D_1}=\varphi_{D_2}$ for different elements $D_1,D_2\in\mathcal{D}_{G,X}(\alpha)$.
(2) There is a subset $\Sigma\subset\Pi$ of type $B_k, k\geqslant 2$, such that $\alpha=\alpha_1+\ldots+\alpha_k$ and $\mathcal{D}_{G,X}(\alpha_i)=\varnothing$ for any $i>1$.
(3) There is a subset $\Sigma\subset\Pi$ of type G_2 such that $\alpha=2\alpha_2+\alpha_1$.

Here α_i denote the simple roots of B_k,G_2 such that α_k (for B_k) and α_2 (for G_2) are short.

The following proposition is a direct corollary of Theorem 2 from [Lo2].

Proposition 2.7. X *is rigid iff there are no distinguished elements in* $\Psi_{G,X}$.

Now let $\Pi'\subset\Pi$ and $\alpha\in\Psi_{G_{\Pi'},X_{\Pi'}}$. If α is distinguished in $\Psi_{G,X}$, then it is distinguished in $\Psi_{G_{\Pi'},X_{\Pi'}}$. The converse is not true: any $\alpha\in\Pi\cap\Psi_{G,X}$ is distinguished in Ψ_{G_α,X_α}. However, if α is of type 2,3 in $\Psi_{G_{\Pi'},X_{\Pi'}}$, then it is of the same type in $\Psi_{G,X}$.

3. Proof of Theorem 1

In this section X is a rigid wonderful G-variety. Put $\mathfrak{h}=\delta_X(x)$ for some point $x\in X^\varnothing$. Let Π^a denote the subset of Π consisting of all roots of type a) for X.

The following two assertions were proved in [Lu3], Section 3.

Lemma 3.1. *If the restriction of* $d_x\delta_X$ *to the* T*-eigenspace* $(T_zX)_\gamma$ *of weight* γ *is injective for any* $\gamma\in\mathfrak{X}(T)$, *then* δ_X *is an isomorphism.*

If $\gamma\not\in\Psi_{G,X}$, then $(T_zX)_\gamma\subset T_z(Gz)$. Since $\delta_X: X\to\overline{G\mathfrak{h}}$ is the normalization morphism, the restriction of δ_X to Gz is an embedding.

Proposition 3.2. *Let* $\alpha\in\Psi_{G,X}$ *and* Π' *be a subset of* Π *containing* $\operatorname{Supp}(\alpha)\cup\Pi^a$. *The restriction of* $d_z\delta_X$ *to* $(T_zX)_\alpha$ *is injective provided so is the restriction of* $d_z\delta_{X_{\Pi'}}$ *to* $(T_zX_{\Pi'})_\alpha$.

Definition 3.3. Let $\alpha\in\Psi_{G,X}$. We say that X is *critical for* α if α is not distinguished in $\Psi_{G,X}$ but is distinguished in $\Psi_{G_{\Pi'},X_{\Pi'}}$ for any $\Pi'\subsetneq\Pi$ containing $\Pi^a\cup\operatorname{Supp}(\alpha)$.

So we need to prove the following claim:

(*) If X is critical for $\alpha\in\Psi_{G,X}$, then the restriction of $d_z\delta_X$ to $(T_zX)_\alpha$ is injective.

Proposition 3.4. *Let X be critical for $\alpha\in\Psi_{G,X}$. (*) holds for X provided $\alpha\not\in\Pi$.*

Proof. It follows from [P], Theorem 3.4, that there is a simple module V and a G-equivariant morphism $\varphi:X\to\mathbb{P}(V)$ such that the restriction of φ to $\overline{X}^\alpha$ is an embedding. So it remains to show that there is a morphism $\psi:\delta_X(\overline{X}^\alpha)\to\mathbb{P}(V)$ such that $\psi\circ\delta_X|_{\overline{X}^\alpha}=\varphi|_{\overline{X}^\alpha}$. Set $Y:=\delta_X(\overline{X}^\alpha),\mathfrak{h}_1:=\delta_X(x)$ for some $x\in X^\alpha,\mathfrak{h}_0:=\delta_X(z)$.

Suppose, at first, that the character group of $G_x,x\in X^\alpha$, is finite. It follows that any G_x-semiinvariant vector in V is $\mathfrak{g}_x$-invariant. Therefore $\dim V^{\mathfrak{f}}\geqslant 1$ for any $\mathfrak{f}\in Y$. By Lemma 2.5, $\mathfrak{f}$ is spherical for any $\mathfrak{f}\in Y$ whence $\dim V^{\mathfrak{f}}\leqslant 1$. Thus the map $\psi:Y\to\mathbb{P}(V),\mathfrak{f}\mapsto V^{\mathfrak{f}}$, is well-defined. Let us check that this map is a morphism of varieties. Let Z denote the subvariety of $\mathfrak{g}^d, d=\dim\mathfrak{h}_1$, consisting of all linearly independent d-tuples and $\pi:Z\twoheadrightarrow \mathrm{Gr}_d(\mathfrak{g})$ be the natural projection. Choose a basis $v^1,\ldots,v^n\in V^*$. Since $\dim V^{\mathfrak{f}}=1$ for all $\mathfrak{f}\in Y$, we have the natural morphism $\widetilde{\psi}:\pi^{-1}(Y)\to\mathbb{P}(V)\cong\mathrm{Gr}_{\dim V-1}(V^*)$ mapping $(\xi_1,\ldots,\xi_d)$ to the linear span of $\xi_iv^j, i=\overline{1,d}, j=\overline{1,n}$. Now recall that $\pi:\pi^{-1}(Y)\to Y$ is the quotient morphism for the natural action $\mathrm{GL}_d:\pi^{-1}(Y)$ and $\widetilde{\psi}:\pi^{-1}(Y)\to\mathbb{P}(V)$ is GL_d-invariant. Therefore $\widetilde{\psi}$ factors through a unique morphism $\psi:Y\to\mathbb{P}(V)$. Clearly, $\psi\circ\delta_X|_{\overline{X}^\alpha}=\varphi|_{\overline{X}^\alpha}$.

Now consider the general case. Since X is critical for α, we have $\Pi=\mathrm{Supp}(\alpha)\cup\Pi^a$. By Lemma 6.1 from [Lo1], $\Pi=\mathrm{Supp}(\alpha)$. Then, inspecting Table 1 in [W], we see that $(\mathfrak{g},\mathfrak{h}_1,\alpha)$ is one of the following triples:

(1) $\mathfrak{g}=\mathfrak{sl}_{n+1}, n\geqslant 2,\mathfrak{h}_1=\mathfrak{gl}_n,\alpha=\alpha_1+\ldots+\alpha_n$.
(2) $\mathfrak{g}=\mathfrak{so}_{2n+1}, n\geqslant 2,\mathfrak{h}_1=\mathfrak{gl}_n\ltimes\bigwedge^2\mathbb{K}^2$, $\alpha=\alpha_1+\ldots+\alpha_n$.
(3) $\mathfrak{g}=\mathfrak{sp}_{2n}, n\geqslant 2,\mathfrak{h}_1=(\mathfrak{sp}_{2n-2}\times\mathfrak{so}_2)\ltimes\mathbb{K},\alpha=\alpha_1+2\alpha_2+\ldots+2\alpha_{n-1}+\alpha_n$.
(4) $\mathfrak{g}=G_2,\mathfrak{h}_1=(\mathfrak{t}_1\times\mathfrak{sl}_2)\ltimes(\mathbb{K}^2\oplus\mathbb{K})$, where $\mathfrak{t}_1$ is a one-dimensional reductive subalgebra of $\mathfrak{g}$, $\alpha=\alpha_1+\alpha_2$.

Note that $\mathrm{codim}_{\mathfrak{h}_1}[\mathfrak{h}_1,\mathfrak{h}_1]=1$ and $[\mathfrak{h}_1,\mathfrak{h}_1]$ is a spherical subalgebra of $\mathfrak{g}$. In all cases $\mathfrak{h}_0$ is the kernel of α in a certain parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}^-$. Since $\alpha\not\in\Pi$, we get $\mathrm{codim}_{\mathfrak{h}_0}[\mathfrak{h}_0,\mathfrak{h}_0]=1$. Analogously to the previous paragraph, the map $\widetilde{\psi}:\overline{G[\mathfrak{h}_1,\mathfrak{h}_1]}\to\mathbb{P}(V)$ mapping $\mathfrak{f}\in\overline{G[\mathfrak{h}_1,\mathfrak{h}_1]}$ to $V^{\mathfrak{f}}$ is a morphism. Since $\mathrm{codim}_{\mathfrak{h}_0}[\mathfrak{h}_0,\mathfrak{h}_0]=\mathrm{codim}_{\mathfrak{h}_1}[\mathfrak{h}_1,\mathfrak{h}_1]$, we see that there is a (unique) G-equivariant map $\iota:Y\to\overline{G[\mathfrak{h}_1,\mathfrak{h}_1]}$ mapping $\mathfrak{h}_1$ to $[\mathfrak{h}_1,\mathfrak{h}_1]$ and $\mathfrak{h}_0$ to $[\mathfrak{h}_0,\mathfrak{h}_0]$. Using a technique similar to that from the previous paragraph, we get that ι is a morphism. It remains to put $\psi=\widetilde{\psi}\circ\iota$. $\square$

The idea to use results of [P] in the proof of smoothness of Demazure's embedding is due to Knop.

So it remains to consider the case when X is critical for $\alpha\in\Pi\cap\Psi_{G,X}$. Suppose at first that $\mathrm{rk}_G(X)=2$. Let D^+,D^- denote different elements of $\mathcal{D}_{G,X}(\alpha)$ and β a unique element of $\Psi_{G,X}\setminus\{\alpha\}$. Since α is not distinguished, we have $\langle\varphi_{D^+},\beta\rangle\neq\langle\varphi_{D^-},\beta\rangle$. It was essentially proved by Luna in [Lu3], Assertion 4, that in this case the restriction of $d_z\delta_X$ to $(T_zX)_\alpha$ is injective (note that $c.[X^\gamma]$ in (**) in the proof of Assertion 4 is equal to $\langle\varphi_{D^\pm},\gamma\rangle$, thanks to Lemmas 3.2.3,3.3 from [Lu1]).

We are going to reduce the general case to the previous one. Choose $\beta \in \Psi_{G,X}$ with $\langle \beta, \varphi_{D^+} \rangle \neq \langle \beta, \varphi_{D^-} \rangle$. Then $\langle \varphi_{D_0^\pm}, \beta \rangle = \langle \varphi_{D^\pm}, \beta \rangle$ for different elements $D_0^\pm \in \mathcal{D}_{G,\overline{X}^{\{\alpha,\beta\}}}(\alpha)$ (see [Lu2], the last paragraph of Subsection 3.2). Clearly, $(T_z X)_\alpha = (T_z \overline{X}^\alpha)_\alpha = (T_z \overline{X}^{\{\alpha,\beta\}})_\alpha$. Choose $x \in X^\alpha$ and put $\mathfrak{h}_0 = \delta_X(x), \widetilde{\mathfrak{h}}_0 = \delta_{\overline{X}^{\{\alpha,\beta\}}}(x)$. It follows from the properties of the Demazure morphisms quoted in Section 2 that $\mathfrak{h}_0$ is an ideal of $\widetilde{\mathfrak{h}}_0$ and $\widetilde{\mathfrak{h}}_0/\mathfrak{h}_0$ is a commutative diagonalizable Lie algebra (of dimension $\mathrm{rk}_G(X) - 2$).

Let Q denote the parabolic subgroups of G containing B^- corresponding to the subset $\Pi^a \sqcup \{\alpha\} \subset \Pi$. Let $\mathfrak{q}_0$ denote a unique ideal in $\mathfrak{q}$ complimentary to $\mathfrak{g}_\alpha$ and Q_0 be the connected subgroup of Q with Lie algebra $\mathfrak{q}_0$. Then $\overline{X}^\alpha$ is G-equivariantly isomorhic to $G *_Q X_\alpha$, where Q acts on X_α via the projection $Q \twoheadrightarrow Q/Q_0$. As a G_α-variety X_α is isomoprhic to $\mathbb{P}^1 \times \mathbb{P}^1$. Note that $\mathfrak{h}_0, \widetilde{\mathfrak{h}}_0$ are ideals in $\mathfrak{g}_x$ of codimension $\mathrm{rk}_G(X), 2$, respectively.

Since $\delta_{\overline{X}^{\{\alpha,\beta\}}}$ is an isomorphism, we have $G_x = N_G(\widetilde{\mathfrak{h}}_0)$. Let $\widetilde{H}_0, H_0$ denote the connected subgroups of G with Lie algebras $\widetilde{\mathfrak{h}}_0, \mathfrak{h}_0$. From Lemma 2.5 it follows that $\mathfrak{h}_0$ is a spherical subalgebra of $\mathfrak{g}$. It follows that $N_G(\mathfrak{h}_0)/H_0$ is a commutative group. In particular, $\widetilde{H}_0/H_0$ commutes with $N_G(\mathfrak{h}_0)/H_0$ whence $N_G(\mathfrak{h}_0) \subset N_G(\widetilde{\mathfrak{h}}_0) = G_x$. On the other hand, $G_x \subset N_G(\mathfrak{h}_0)$, for δ_X is G-equivariant. So the restriction of δ_X to X^α is injective.

Now we apply the argument from [Lu3], proof of Assertion 4. Choose $x \in X_\alpha^\varnothing$. By above, $\mathfrak{g}_x = \mathfrak{t} + \mathfrak{q}_0$. Let $\mathfrak{m}$ denote a unique Levi subalgebra of $\mathfrak{q}$ containing $\mathfrak{t}$. Set $\mathfrak{m}_0 := \mathfrak{m} \cap \mathfrak{q}_0$. Since $\mathfrak{g}_x \subset \mathfrak{n}_\mathfrak{g}(\delta_X(x))$ and $\mathfrak{g}_x/\delta_X(x)$ is a diagonalizable Lie algebra, we see that $\underline{\mathfrak{q}} := \mathrm{R}_u(\mathfrak{q}) + [\mathfrak{m}_0, \mathfrak{m}_0] \subset \delta_X(x)$. So we may consider $\delta_X|_{X_\alpha}$ as a morphism to $\mathfrak{q}/\underline{\mathfrak{q}}$. The Lie algebra $\mathfrak{q}/\underline{\mathfrak{q}}$ is identified with $\mathfrak{m}_1 := \mathfrak{g}_\alpha \oplus \mathfrak{z}(\mathfrak{m}_0)$. Let M_1 denote the connected subgroup of G. As we have shown above, $N_{M_1}(\delta_X(x))$ is a maximal torus of M_1 for any $x \in X_\alpha^\varnothing$. Analogously to the proof of Assertion 4 in [Lu3], $\overline{M_1 \delta_X(x)}$ is smooth. So δ_X is an isomorphism of X_α to $\overline{M_1 \delta_X(x)}$ whence its restriction to $(T_x X)_\alpha$ is injective.

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